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ON FINITE PRODUCTS OF SOLUBLE GROUPS

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ABSTRACT

Let the finite group $G = AB$ be the product of two soluble subgroups A and B, and let π be a set of primes. We investigate under which conditions for the maximal normal π-subgroups of A, B and G the following holds: $O_\pi(G) \cap O_\pi(G) \subseteq O_\pi(G)$.

1. Introduction

A result of Johnson [5] says that for every finite soluble group $G = AB$ which is the product of two subgroups A and B and for every set of primes π, the maximal normal π-subgroups satisfy $O_\pi(A) \cap O_\pi(B) \subseteq O_\pi(G)$. It is natural to ask whether this extends to arbitrary finite products of groups and, in particular, to products of finite soluble groups.

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Some examples which show that this question has a negative answer in general will be given in the last section. Groups of the following type turn out to be crucial.

A group G is said to be of type $\Gamma(q)$ if there exists a power $q = p^m$ of some odd prime p such that G is a subgroup of the group $\mathrm{P\Gamma L}_2(q)$ and G contains a normal subgroup isomorphic to $L_2(q)$ such that the index $m = |G: L_2(q)| \equiv 1 \bmod 2$.

THEOREM 1.1: *Let the finite group $G = AB$ be the product of two soluble subgroups A and B. If G does not involve any section of type $\Gamma(q)$ for odd numbers m and q, then for the maximal normal subgroups of order we have $O(A) \cap O(B) \subseteq O(G)$.*

COROLLARY 1.2: *If the finite group $G = AB$ is the product of two soluble subgroups A and B and if no section of type $\Gamma(q)$ for odd numbers m and q is involved in G, then $O_\pi(A) \cap O_\pi(B) \subseteq O_\pi(G)$ for every set of primes π not containing the prime 2.*

We will frequently use a result of Kazarin [6] by which the groups that can occur as non-abelian composition factors of a product of two finite soluble subgroups are known.

The notation is standard and can be found, for instance, in [3]. $\mathrm{P\Gamma L}_2(q)$ denotes the automorphism group of $L_2(q) = \mathrm{PSL}(2,q)$. All groups considered are finite.

2. Preliminaries

By Kazarin [6], the groups which may occur as composition factors of a product of two finite soluble groups are contained in the following set:

$$\mathfrak{M} = \{L_2(q), q > 3; L_3(q), q < 9; L_4(2), M_{11}, \mathrm{PSp}_4(3), U_3(8)\}.$$

The following table contains relevant information on the orders of these groups and their outer automorphism groups.

Group G	Order of G	$\lvert \operatorname{Out} G\rvert$
$U_3(8)$	$19.3^4.7.2^6$	2.3^2
$\mathrm{PSp}_4(3)$	$2^6.3^4.5$	2
$L_4(2)$	$2^6.3^2.5.7$	2
M_{11}	$2^4.3^2.5.11$	1
$L_3(7)$	$2^5.3^2.7^3.19$	2.3
$L_3(3)$	$3^3.2^4.13$	2
$L_3(4)$	$2^6.3^2.5.7$	$2^2.3$
$L_3(5)$	$2^5.3.5^3.31$	2
$L_3(8)$	$2^9.3^2.7^2.73$	2.3
$L_2(q)$, $q=p^n$	$\epsilon q(q^2-1)$, $\epsilon=(p-1,2)^{-1}$	$\epsilon^{-1}n$

We recall the following well-known result of Dickson which can be found, for instance, in [3], II, 8.27.

LEMMA 2.1: *Let H be a maximal subgroup of the simple group $X \simeq L_2(q)$, where $q=p^n$ for a prime p. Then one of the following conditions holds:*

(1) *$H \simeq D_m$, a dihedral group of order $m=(q\pm 1)\epsilon$ with $\epsilon=(q-1,2)^{-1}$,*
(2) *$H \simeq A_4$ or S_4,*
(3) *$H \simeq A_5$ for $p=5$ or $p^{2n}-1 \equiv 0 \bmod 5$,*
(4) *$H \simeq E \rtimes C$, where E is an elementary-abelian group of order p^n and C is a cyclic group of order $\epsilon(q-1)$,*
(5) *$H \simeq L_2(p^s)$ where s divides n, or $H \simeq \mathrm{PGL}_2(p^s)$ where $2s$ divides n.*

Lemma 2.1 has the following consequence. Note that the second statement becomes false when the prime r is not odd.

LEMMA 2.2: *All local subgroups of a simple group $X \simeq L_2(q)$ are soluble. Furthermore, if r is an odd prime and Y and T are maximal r-local subgroups of X, then Y and T are conjugate in X.*

The next result generalizes a theorem of Itô [4] on the factorizations of the linear fractional groups.

LEMMA 2.3: *Let A and B be maximal subgroups of a simple group $N \simeq L_2(q)$, where $q=p^n$ for a prime p. If the order of N divides $\lvert A\rvert\lvert B\rvert\lvert \operatorname{Out} N\rvert$, then one of the following conditions holds:*

(1) *$q=4$ or 5, and $\{\lvert A\rvert,\lvert B\rvert\}=\{12,10\}$ or $\{6,10\}$,*
(2) *$q=7$, $\{\lvert A\rvert,\lvert B\rvert\}=\{21,24\}$,*
(3) *$q=9$, $\{\lvert A\rvert,\lvert B\rvert\}=\{10,36\},\{24,60\},\{36,60\}$ or $\{60,60\}$,*
(4) *$q=11$, $\{\lvert A\rvert,\lvert B\rvert\}=\{12,55\}$ or $\{55,60\}$,*
(5) *$q=19$, $\{\lvert A\rvert,\lvert B\rvert\}=\{19.9,60\}$ or $\{19.9,20\}$,*

(6) $q = 29$, $\{|A|, |B|\} = \{29.14, 60\}$ *or* $\{29.14, 30\}$,

(7) $q = 59$, $\{|A|, |B|\} = \{59.29, 60\}$,

(8) $q = 119$, $\{|A|, |B|\} = \{119.59, 60\}$ *or* $\{119.59, 120\}$,

(9) $\{|A|, |B|\} = \{\epsilon q(q-1), \nu(q+1)\}$ *where* $\epsilon = (2, q-1)^{-1}$, $\nu = (2, q)$, *and the subgroup of order* $\nu(q+1)$ *is dihedral,*

(10) $q = 23$, $\{|A|, |B|\} = \{23.11, 24\}$,

(11) $q = 16$, $\{|A|, |B|\} = \{16.15, 2.17\}$ *or* $\{60, 34\}$.

Here — with the exeption of case (7) *— every subgroup of order* 60 *is a group of type* $L_2(4)$.

Proof: Set $\epsilon = (p-1, 2)^{-1}$. Then

$$(*) \qquad |N| = \epsilon p^n(p^{2n} - 1) \text{ divides } |A||B|2n,$$

and it is easy to see that $|A||B| \equiv 0 \bmod p$. Hence we may assume that $|A| \equiv 0 \bmod p$.

If $A \simeq A_4$, then the maximality of A implies that the Sylow-2-subgroup of N is a four-group. If q is even, then $q = 4$ and $N \simeq A_5 \simeq L_2(4) \simeq L_2(5)$. Therefore the first statement holds.

Let $p = 3$. Since $|N| \leq |A||B|2n$, we have that $|N\colon B| \leq 24 \log_3 q$. Now it follows easily from Lemma 2.1 that $|N\colon B| \geq q$ for each proper subgroup B of the group $N \simeq L_2(q)$ with $q > 9$. Hence $q \leq 24 \log_3 q$. Using condition $(*)$ and Lemma 2.1 this possibility can be excluded ($q \leq 3^4$).

Let $A \simeq S_4$. Then $q \equiv 1 \bmod 2$. Hence $p = 3$. By $(*)$ we see that $|N| \leq |A||B|2n$ and $q \leq |N\colon B| \leq 48 \log_3 q$. Then $q \leq 3^4$. A case-by-case analysis using Lemma 2.1 and condition $(*)$ gives $q = 9$ and $\{|A|, |B|\} = \{24, 60\}$.

Assume further that $A \simeq A_5$. If $p = 2$ then $q \leq |N\colon B| \leq 60 \log_2 q$ and $q \leq 2^9$. A case-by-case analysis shows that the only possibility in this case is $q = 2^4$ and $|B| = 34$. Note that here a group $L = \mathrm{P\Gamma L}_2(16)$ has the interesting factorization $L = UV$ with $U \cap N = A$, $V \cap N = B$ and $U \cap V = 1$. We also have $|A| = 60$ and $|B| = 34$.

Let $p = 3$. Then $q \leq 120 \log_3 q$ and $q < 3^6$. It is easy to see that in this case $q = 9$ and B is a maximal subgroup whose order is divisible by 3. Hence $\{|A|, |B|\} = \{60, 36\}, \{60, 60\}, \{60, 24\}$.

Assume next that $p = 5$. Then $q \leq 120 \log_5 q$ and $q \leq 5^3$. Using Lemma 2.1 and condition $(*)$ we obtain a contradiction.

Now we may suppose that $|A| = q(q-1)\epsilon$ or $|A| = p^s(p^{2s} - 1)\epsilon$, where s divides n or $|A| = p^s(p^{2s} - 1)$ with $2s$ dividing n. In this case $s \leq n/2$ and $|B| \equiv 0 \bmod p$.

Since the cases A_4, A_5 and S_4 were considered above we may suppose that $|B|$ divides $p^t(p^{2t}-1)$, where t divides n or $|B| = q(q-1)\epsilon$.

Let $|B|$ be a divisor of $p^t(p^{2t}-1)$ with $t < n$. Then

$$(**) \qquad \epsilon p^n(p^{2n}-1) \text{ divides } p^s(p^{2s}-1)p^t(p^{2t}-1)n\mu \text{ with } \mu = (2, p-1).$$

If $n = p^\lambda r$ with $(r,p) = 1$, then $n \leq s + t + \lambda$ where $\lambda \leq \log_p n$.

If $s, t \leq n/3$ then $n = p = 3$. Using conditions $(*)$ and $(**)$ we obtain a contradiction by Lemma 2.1. Hence we may assume that $s = n/2$. If $t \leq n/3$ then $n = 6$ and $\epsilon(p^{12}-1)$ divides $(p^6-1)(p^4-1).12$. It is not difficult to see that this is impossible. Hence $t = n/2$ and $\epsilon p^{2t}(p^{4t}-1)$ divides $p^{2t}(p^{2t}-1)^2 2t\mu$. This leads to the following:

$$p^{2t}+1 \text{ divides } (p^{2t}-1)8t.$$

If $p = 2$ then $2^{2t}+1$ divides $(2^{2t}-1)t$. However, $(2^{2t}+1, 2^{2t}-1) = 1$ and $2^{2m}+1$ divides m, a contradiction.

If $p \neq 2$ then it is easy to see that $(p^{2t}+1, 4) = 2$ and $(p^{2t}-1, p^{2t}+1) = 2$. Hence $p^{2t}+1$ divides $2t$ which also gives a contradiction.

Now we may assume that in each case $|A| = \epsilon q(q-1)$. Then

$$(***) \qquad q+1 \text{ divides } |B|n\mu, \quad \text{where } \mu = (p-1, 2).$$

If B is as in Lemma 2.1.5, then $p^{ms}+1$ divides $(p^{2s}-1)2ms$.

If $m = 2$, then $(p^{ms}+1, p^{2s}-1) = 2$ and $p^{ms}+1$ divides $4ms$. Then $ms \leq 4$ and it is easy to see that this is not the case. If $m > 2$, then $p^{(m-2)s} \leq 2ms$ and $(m-2)s \leq 5$. It is not difficult to show that this contradicts condition $(*)$.

Hence either $|B| = (q+1)\mu$ with $\mu = (2, p-1)$ or $|B| \in \{12, 24, 60\}$.

Let $|B| = 12$. Then $q+1$ divides $24n$ and $n \leq 7$. If $p = 2$, then even 2^n+1 divides $3n$ and $n = 3$. Now $|B| = (q+1)2 = 18$ and $\{|A|, |B|\}$ is as in (9). If $p = 3$ then $n \leq 4$, and it is easy to see that this is impossible by Lemma 2.1 and condition $(***)$. If $p = 5$ then $n \leq 2$, and it follows that $N \simeq L_2(5)$ and $\{|A|, |B|\} = \{10, 6\}$ or $\{10, 12\}$. If $p \geq 7$ then $n = 1$ and $p+1$ divides 24, so that $p \leq 23$.

Let $|B| = 24$. Then $q+1$ divides $48n$. As above it follows that $n \leq 8$ and, if $p = 2$, then $q = 8$. If $p = 3$ then $q \leq 4$, which is not the case. If $p = 5$ then $q \leq 5^3$. It is not difficult to see that this also leads to a contradiction. If $p \geq 7$, then $n = 1$ and $p + 1 \leq 48$ implies that either $\{|A|, |B|\} = \{\epsilon q(q-1), \nu(q+1)\}$ or $q = 7$ or 11.

Finally, let $|B| = 60$. Then $q+1$ divides $120n$. If q is even then $q+1$ divides $15n$. It is easy to show that this leads to a contradiction. Hence q is odd. If $p = 3$, then $n \leq 4$. An easy calculation excludes this case. (An exception is $q = 9$ and $\{|A|, |B|\} = \{36, 60\}$.) If $p = 7$ then we also obtain a contradiction. If $p = 11$ also $q = 11$ and $\{|A|, |B|\} = \{55, 60\}$. The cases $p = 13$ or 17 do not occur. If $p = 19$ then $q = 19$ and $\{|A|, |B|\} = \{19.9, 60\}$. If $p > 19$ then $q = p$ and $q = 19, 29, 59, 119$.

Hence in all cases we have $|B| = (q+1)\nu$ with $\nu = (2, p)$ as stated in (9).

COROLLARY 2.4: *Let A and B be two maximal soluble subgroups of a simple group $N = L_2(q)$ with $q = p^n$ for some prime p. If the order of N divides $|A||B||\operatorname{Out} N|$ then one of the following statements holds:*

(1) $\{|A|, |B|\} = \{\epsilon q(q-1), \nu(q+1)\}$ *with* $\epsilon = (2, q-1)^{-1}$ *and* $\nu = (2, q)$,
(2) $q = 4$ *or* 5 *and* $\{|A|, |B|\} = \{12, 10\}$ *or* $\{6, 10\}$,
(3) $q = 7$ *and* $\{|A|, |B|\} = \{21, 24\}$,
(4) $q = 11$ *and* $\{|A|, |B|\} = \{24, 55\}$,
(5) $q = 23$ *and* $\{|A|, |B|\} = \{23.11, 24\}$.

LEMMA 2.5: *Let $N \in \mathfrak{M}$ be a non-abelian composition factor of a finite group G which is the product of two of its soluble subgroups. Then there exist maximal soluble subgroups A and B of N such that the order of N divides $|A||B||\operatorname{Out} N|$ and one of the following conditions holds:*

(1) $N \simeq U_3(8)$ *and* $\{|A|, |B|\} = \{19.3, 2^6.7.3\}$,
(2) $N \simeq L_3(3)$ *and* $\{|A|, |B|\} = \{13.3, 3^3.2^4\}$,
(3) $N \simeq L_3(5)$ *and* $\{|A|, |B|\} = \{31.3, 2^4.5^3\}$,
(4) $N \simeq L_3(8)$ *and* $\{|A|, |B|\} = \{73.3, 2^6.7^2\}$,
(5) $N \simeq \mathrm{PSp}_4(3)$ *and* $\{|A|, |B|\} = \{2^5.5, 3^4.2^4\}$,
(6) $N \simeq L_4(2)$ *and* $\{|A|, |B|\} = \{2^3.7.3, 2^2.3.5\}$,
(7) $N \simeq M_{11}$ *and* $\{|A|, |B|\} = \{55, 2^4.3^2\}$,
(8) $N \simeq L_2(q)$ *and* $\{|A|, |B|\}$ *is as in the preceding corollary.*

Proof: The first statement follows from Lemma 2.5 of [6]. Now we will consider the groups in $\mathfrak{M}$ separately. In each case A and B are maximal subgroups of the group N with the following property:

(∗) The order of N divides $|A||B||\operatorname{Out} N|$.

(1) $N \simeq U_3(8)$.

In this case the maximal soluble subgroup of N which contains an element of order 19 has order 19.3 (see Lemma 5 of [8]). Hence we may assume that

$|A| = 19.3$. Now $2^6.3^4.7.19$ divides $3.19.|B|.3^2.2$. Therefore $|B| \equiv 0 \mod 2^5.7.3$. It is easy to see that $O_2(B) \neq 1$ and B is contained in a maximal parabolic subgroup of N which coincides with its Borel subgroup (since N has Lie rank 1). The assertion is proved.

(2) $N \simeq L_3(3)$.

We may now assume that $|A| = 13.3$ (see Lemma 2 of [8]). Then $|B| \equiv 0 \mod 3^2.2^4$. It is easy to see that B is a maximal parabolic subgroup of N and its order is $3^3.2^4$.

(3) $N \simeq L_3(5)$.

Again by Lemma 2 of [8] we may assume that $|A| = 31.3$. By (*) $2^4.5^3$ divides $|B|$. By Lemma 1.10 of [6] the subgroup B is contained in a parabolic subgroup P of N. By considering the structure of the factor group $P/O_5(P)$ it is easy to prove that B must have order $2^4.5^3$.

(4) $N \simeq L_3(8)$.

As in the first two cases we see that $|A| = 73.3$. Then $2^8.7^2$ divides the order of B by condition (*). By Lemma 1.10 of [6] the group B is contained in a parabolic subgroup P of N. Considering the structure of the factor group $P/O_2(P)$ we can show that B coincides with a Borel subgroup of N. This gives the required statement.

(5) $N \simeq \mathrm{PSp}_4(3)$.

Without loss of generality we may assume that 5 divides $|A|$. It follows from 5.1.7 of [9] that the maximal subgroup X of N containing A is an extension of an elementary abelian group of order 16 by A_5. By condition (*), 27 divides $|B|$ and B must be a parabolic subgroup of N of order $3^4.2^4$. The maximal soluble subgroup of X which contains an element of order 5 has order $2^5.5$.

(6) $N \simeq L_4(2)$.

Let 7 be a divisor of $|A|$. It is easy to see that $|A| = 2^3.7.3$ (as a soluble subgroup of $\mathrm{AGL}(3,2) \subseteq \mathrm{GL}_4(2)$) or $|A| = 7.6$. Since $2^6.3^2.5.7$ divides $|A||B|2$, then either $2^2.3.5$ divides $|B|$ or $2^4.3.5$ divides $|B|$. The second case is impossible for a soluble subgroup $B \subseteq L_4(2) \simeq A_8$. The first case is realized with $2^2.3.5 = |A|$.

(7) $N \simeq M_{11}$.

This is a consequence of the fact that all subgroups of the group M_{11} are known.

(8) $N \simeq L_2(q)$.

The required statement is already contained in Corollary 2.4.

(9) $N \simeq L_3(7)$.

Assume that such a group appears as a composition factor of a product of two soluble groups. Then it follows from Lemma 2.5 of [6] that a group of type $L_3(7)$ must contain soluble subgroups A and B having property $(*)$. Using Lemma 2 of [8] we may assume that $|A| = 19.3$. Then $2^4.7^3$ divides $|B|$. By Lemma 1.10 of [6] the subgroup B is contained in a parabolic subgroup P of a group N. The Borel subgroup of a group N has order dividing $7^3.6^2$. On the other hand, the factor group $P/O_7(P)$ is an extension of a group of type $L_2(7)$ by a group of order 12. It is easy to see that P does not contain a soluble subgroup of order $2^4.7^3.m$, where $m \geq 1$ is an integer. This is a contradiction.

(10) $N \simeq L_3(4)$.

This case can be excluded in a similar way as above.

COROLLARY 2.6: *Let $N \in \mathfrak{M}$ be a simple group having maximal soluble subgroups A and B such that the order of N divides $|A||B||\operatorname{Out} N|$. Then either $(|O(A)|, |O(B)|) = 1$ or $N \simeq L_3(3)$ and $O(A) \cap O(B) = 1$.*

3. Proof of the theorem

The proof of the theorem is divided into a series of steps. Assume that the theorem is false, and let $G = \tilde{A}\tilde{B}$ be a counterexample with minimal order, where $\tilde{A}$ and $\tilde{B}$ are soluble subgroups of G.

LEMMA 3.1: *The group G has a unique minimal normal subgroup N and the maximal normal soluble subgroup $S(G)$ of G is trivial. In particular $C_G(N) = 1$.*

Proof: Assume there exist two different minimal normal subgroups N and M of G, and put $K/M = O(G/M)$ and $L/N = O(G/N)$. By the minimality of G it follows obviously that $O(A) \cap O(B)$ is contained in $K \cap L = O(G)$, and this contradiction shows that there exists a unique minimal normal subgroup N of G.

Assume that $S = S(G) \neq 1$. Then $G/S = \bar{G} = \bar{A}\bar{B}$, where $\bar{A} = \tilde{A}S/S$ and $\bar{B} = \tilde{B}S/S$. Since G is a minimal counterexample, $O(\bar{A}) \cap O(\bar{B}) \subseteq O(\bar{G}) = 1$. Hence $O(\tilde{A}) \cap O(\tilde{B}) \subseteq S(G)$. Put $H = S\tilde{A}$. If $H = G$, then G is a soluble and Johnson's result in [5] implies that $O(\tilde{A}) \cap O(\tilde{B}) \subseteq O(G)$. If $H \neq G$, then $H = \tilde{A}(H \cap \tilde{B})$ and $O(\tilde{A}) \cap O(\tilde{B}) \subseteq O(\tilde{A}) \cap (O(H \cap \tilde{B})) \subseteq O(H)$ by the minimality of G. Since $O(\tilde{A}) \cap O(\tilde{B}) \subseteq S$, then $O(\tilde{A}) \cap O(\tilde{B}) \subseteq S \cap O(H) \subseteq O(S) \subseteq O(G)$, a contradiction. Hence $S = S(G) = 1$, and the lemma is proved.

LEMMA 3.2: *Let $D = O(\tilde{A}) \cap O(\tilde{B})$. Then $N_G(D) = N_{\tilde{A}}(D)N_{\tilde{B}}(D)$.*

Proof: This follows from [1], Lemma 1.2.2.

LEMMA 3.3: *$N\tilde{A} = N\tilde{B} = G = \tilde{A}\tilde{B}$.*

Proof: Assume that $K = N\tilde{A} \neq G$. Since $D \subseteq \tilde{A} \subseteq K = \tilde{A}(K \cap \tilde{B})$ and $|K| < |G|$, we have $D \subseteq O(K)$. Then $D \subseteq \tilde{A} \subseteq K = \tilde{A}(K \cap \tilde{B})$. Since $|K| < |G|$ we have $D \subseteq O(K)$. Then

$$[D, N] \subseteq O(K) \cap N \subseteq O(N) \subseteq O(G) = 1,$$

so that $D \subseteq C_G(N) = 1$, a contradiction. Hence $K = N\tilde{A} = G$. Similarly $N\tilde{B} = G = \tilde{A}\tilde{B}$.

The next four results are number-theoretical.

LEMMA 3.4: *Let m, n, k be positive integers such that m^k divides $n^k k!$. Then m divides n.*

Proof: Let p be any prime dividing m, and suppose that p does not divide n. Then $(p, n^k) = 1$ and $p^k | k!$. This is impossible since the highest power of p not dividing $k!$ is p^ν where

$$\nu = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots < \frac{k}{p-1}.$$

An induction argument proves the lemma.

COROLLARY 3.5: *If m^k divides $n^s k!$ where $s \leq k$, then m divides n.*

Proof: Since m^k divides $n^s k!$, it also divides $n^k k!$. Now apply the preceding lemma.

LEMMA 3.6: *If n^{sr-r} divides $(sr)!$, then $s \leq (p-1)/(p-2)$ for any prime $p \geq 3$ dividing n.*

Proof: Let $p \geq 3$ be a divisor of n. Then as above

$$p^{sr-r} \leq p^{\frac{sr}{p} + \frac{sr}{p^2} + \cdots} \leq p^{\frac{sr}{p-1}}.$$

Hence $s \leq (p-1)/(p-2)$.

COROLLARY 3.7: *If $p \geq 5$ divides n in the preceding lemma, then $s = 1$.*

LEMMA 3.8: *The group $N = N_1 \times N_2 \times \cdots \times N_k$ is a product of isomorphic simple groups N_i and $\tilde{A}$, $\tilde{B}$ and G act transitively on the set $\Omega = \{N_1, N_2, \ldots, N_k\}$. If X is a maximal subgroup of G containing $\tilde{A}$ or $\tilde{B}$, then D is contained in $O(X)$ and one of the following conditions holds:*

(i) *There exists a partition* $\mathfrak{P}$ *of the set* $\{1,2,\ldots,k\}$ *into subsets of equal cardinality* $\neq 1$ *such that* $N \cap X = \prod_{f\in\mathfrak{P}} \Delta_f$, *where* Δ_f *is the full diagonal subgroup of* N *with respect to the element* f *of* $\mathfrak{P}$,

(ii) $N \cap X = \prod_{i=1}^{k} X_i$, *where* $X_i = N_i \cap X \simeq X_1$ *for* $i = 1,2,\ldots,k$.

Proof: The transitivity of $\tilde{A}, \tilde{B}$ and G on Ω follows from Lemma 3.1 and Lemma 3.3. The last assertion follows from the Appendix in [2].

LEMMA 3.9: $n^k = |N| = |N_1|^k$ *divides* $|\operatorname{Out} N_1|^k |\tilde{A} \cap N||\tilde{B} \cap N| k!$.

Proof: By Lemma 3.3 we have

$$|\tilde{A}\tilde{B}|/|\tilde{A}\cap\tilde{B}| = |G| = |\tilde{A}||N|/|\tilde{A}\cap N| = |\tilde{B}||N|/|\tilde{B}\cap N|.$$

Then

$$|\tilde{A}||\tilde{B}||N|^2|/|\tilde{A}\cap N||\tilde{B}\cap N| = |G|^2 = |G||\tilde{A}||\tilde{B}|/|\tilde{A}\cap\tilde{B}|.$$

This implies

$$|N||\tilde{A}\cap\tilde{B}| = |G\colon N||N\cap\tilde{A}||N\cap\tilde{B}|.$$

The set G_0 of elements g in G such that $N_i^g = N_i$ for every $i \in \{1,2,\ldots,k\}$ is a normal subgroup of G such that N is normal in G_0. Moreover, $G_0 \subseteq \prod_{i=1}^{k} K_i$ where $N_i \subseteq K_i \subseteq \operatorname{Aut} N_i$. Hence $|N|$ divides $|G/G_0||G_0\colon N||N\cap\tilde{A}||N\cap\tilde{B}|$. Since G/G_0 is a subgroup of the symmetric group S_k acting on Ω, the lemma is proved.

In the following let A and B be maximal subgroups of G containing $\tilde{A}$ and $\tilde{B}$, respectively, and G_0 is the set of all $g \in G$ such that $N_i^g = N_i$ for every $i \in \{1,2,\ldots,k\}$.

LEMMA 3.10: *If* $D \cap N \neq 1$, *then* $|N_1|$ *divides* $|N_1 \cap A||N_1 \cap B||\operatorname{Out} N_1|$, *and there exists a prime* p *in* $\pi(D)$ *such that* $p \in \pi(O(N_1 \cap B)) \cap \pi(O(N_1 \cap A))$.

Proof: Since $D \cap N \neq 1$, also $O(A) \cap O(B) \cap N \neq 1$ and $D \cap N \subseteq O(A) \cap N \subseteq O(A\cap N)$, $D\cap N \subseteq O(B)\cap N \subseteq O(B\cap N)$. By Lemma 3.8 it follows that $N\cap A = \prod_{i=1}^{k} X_i$, where $X_i = N_i \cap A$ with $i = 1,2,\ldots,k$, and $N \cap B = \prod_{i=1}^{k} Y_i$ where $Y_i = (N_i \cap B)$. Moreover, $|N\cap A| = |X_1|^k$, $|N\cap B| = |Y_1|^k$ since A and B permute the X_i and Y_i, respectively. Moreover, $O(A\cap N) = \prod_{i=1}^{k} O(N_i\cap A) = \prod_{i=1}^{k} O(X_i)$ and $O(N \cap B) = \prod_{i=1}^{k} O(Y_i)$. Hence $\pi(D \cap N) \subseteq \pi(O(X_1)) \cap \pi(O(Y_1))$. By Lemma 3.9, $|N_1|^k$ divides $|\operatorname{Out} N_1|^k |X_1|^k |Y_1|^k k!$. By Lemma 3.4, $|N_1|$ divides $|\operatorname{Out} N_1||X_1||Y_1|$.

LEMMA 3.11: $D \cap N = 1$.

Proof: Assume $D \cap N \neq 1$. Then $|N_1|$ divides $|\operatorname{Out} N_1||N_1 \cap A||N_1 \cap B|$, and there exists a prime p in the set $\pi(D) \cap \pi(O(N_1 \cap A)) \cap \pi(O(N_1 \cap B))$. By Kazarin [6] N_1 is one of the groups in $\mathfrak{M} = \{L_2(q), q \geq 3; L_3(q), q < 9; L_4(2), M_{11}, \mathrm{PSp}_4(3), U_3(8)\}$. By Corollary 2.6, $N_1 \simeq L_3(3)$. However in this case $D \cap N = 1$, since $O(N_i \cap A) \cap O(N_i \cap B) = 1$ for each i and $|\operatorname{Out} N_1| = 2$. This leads to a contradiction.

LEMMA 3.12: *If $O(A) \cap N \neq 1$, then $A \cap N = \prod_{i=1}^{k} X_i$ where the $X_i = A \cap N_i$ are conjugate under A and $1 \neq O(X_i) \subseteq O(A)$. If $|X_i| \equiv 0 \bmod 2$, then $N_i^d = N_i$ for each d in D and each $i = 1, 2, \ldots, k$.*

Proof: By Lemma 3.8, either $A \cap N$ is a direct product of subgroups isomorphic to N_1 or $A \cap N = \prod_{i=1}^{k} X_i$ where the $A \cap N_i = X_i$ are conjugate to X_1. Since $O(A) \cap N \subseteq O(A \cap N)$ the first possibility is excluded and we are done. Suppose that $|X_i| \equiv 0 \bmod 2$. Since $[D, A \cap N] \subseteq [O(A), A \cap N] \subseteq O(A) \cap N \subseteq O(A \cap N)$ then $d^{-1}g^{-1}dg = [d, g]$ is of odd order for each element g in $A \cap N$. Suppose that $N_i^d \neq N_i$ for some i and τ is an involution of X_i. Then $N_i^d = N_j$ for some $j \neq i$ and τ^d is an involution of N_j, Since $\tau^d \tau$ is also an involution and $\tau^d \tau = [d, \tau] \in [O(A) \cap N, d]$, this is a contradiction.

LEMMA 3.13: *Let $M = M_1 \times \cdots \times M_t$ be the product of isomorphic simple groups, and let K be the full diagonal subgroup of M. If $H \cap M_i \neq 1$ for each $i \leq t$, then $< H, K >= M$.*

Proof: It is enough to prove that $M_i \subset < H, K >$ for each $i \leq t$. Without loss of generality we may assume that $t = 2$ and $i = 1$. Then $M = M_1 \rtimes K$ and by Dedekind's modular law we have $L =< H, K >= (L \cap M_1) \rtimes K$. Each element g in M can be written as (x, y) where x and y are in M_1 and $K = \{(x, x) | x \in M_1\}$, $M_1 = \{(x, 1) | x \in M_1\}$. If $u = (y, y) \in K$ and $z = (x, 1) \in M_1$, then $z^u = (x^y, 1) \in M_1$ for each $y \in M_1$. Since M_1 is a simple group and $L \cap M_1 \neq 1$ it follows that $L \cap M_1 = M_1$, so that $L = M$.

LEMMA 3.14: *If $N_1 \simeq L_2(q)$ where q is an odd number, then $|G_0\colon N|$ is a power of 2.*

Proof: If this is not so there exists a section of G which is isomorphic to a group of type $\Gamma(q)$.

LEMMA 3.15: $A \cap N \neq 1 \neq B \cap N$.

Proof: Assume that $A \cap N = 1$. By Lemma 3.9 we have that $n^k = |N|$ divides $|\operatorname{Out} N_1|^k |B \cap N| k!$. By Lemma 3.8, either $|B \cap N| = n^s$ where $k = sr$ or

$|B \cap N| = m^k$ where $m = |N_1 \cap B|$. By assumption and the above table we have $|\operatorname{Out}_G(N_i)| = 2^\alpha 3^\beta$ for some α and β. Let n_0 be the $\{2,3\}'$-part of $n = |N_1|$. By Lemma 3.8, either n_0^{sr-s} divides $(sr)!$ or n^k divides $|\operatorname{Out} N_1|^k m^k k!$. In the first case using Lemma 3.6 we obtain that $r \leq (p-1)/(p-2)$ for each prime p in $\pi(n_0)$ and $r \leq 4/3$ which implies $r = 1$, a contradiction to Lemma 3.6. In the second case n divides $|\operatorname{Out} N_1||N_1 \cap B|$ and $|N_1\colon N_1 \cap B| \leq |\operatorname{Out} N_1|$, which is not the case for groups in the class $\mathfrak{M}$.

LEMMA 3.16: *Suppose that $O(A) \cap N = 1$. Let $\{N_{i_1}, \dots, N_{i_s}\}$ be the smallest subset of Ω such that $A \cap (N_{i_1} \times N_{i_2} \times \cdots \times N_{i_s}) \neq 1$. Then there exists a partition $\mathfrak{P}$ of the set $\{1, 2, \dots, k\}$ into equal parts such that the subgroups $\nabla_f = \prod_{j \in f} N_j$ with $f \in \mathfrak{P}$ are permuted by A transitively and $O(A)$ fixes each ∇_f.*

Proof: The hypothesis implies that $[O(A), A \cap N] \subseteq O(A) \cap N = 1$. Let Ω_1 be the set of elements N_i in Ω such that $N_i^a = N_i$ for each a in $O(A)$. It is obvious that $\Omega_1^x = \Omega_1$ for each x in $O(A)$ and, since A is transitive on Ω, then Ω_1 is either empty or $\Omega_1 = \Omega$. In this case $O(A) \subseteq G_0$. If $N_i \cap A \neq 1$ for some $i \leq k$ then $N_i^a \cap N_i \geq N_i \cap A \neq 1$ for each a in $O(A)$. Hence $N_i^a = N_i$ for each a in $O(A)$. Since A acts transitively on Ω we have that $N_j^a = N_j$ for each $j \leq k$ and each a in $O(A)$.

Let $\{N_{i_1}, \dots, N_{i_s}\}$ be the smallest subset of Ω such that

$$E = A \cap (N_{i_1} \times \cdots \times N_{i_s}) \neq 1.$$

Then $E \subseteq (N_{i_1}^a \times \cdots \times N_{i_s}^a) \cap (N_{i_1} \times \cdots \times N_{i_s})$, which is a normal subgroup of N. Hence we may suppose that there exists a smallest subset f of $\{i_1, \dots, i_s\}$ such that $\nabla_f = \prod_{j \in f} N_j$ is fixed by $O(A)$. Since $\nabla_f \cap \nabla_f^x$ is also fixed by $O(A)$ for each x in A and is normal in N, we have either $\nabla_f \cap \nabla_f^x = 1$ or $\nabla_f \cap \nabla_f^x = \nabla_f$ for x in A. Since A acts transitively on Ω the partition $\mathfrak{P}$ with the required properties exists.

LEMMA 3.17: *$O(A) \cap N \neq 1$ or $O(B) \cap N \neq 1$.*

Proof: Suppose that $O(A) \cap N = 1 = O(B) \cap N$.

1. FIRST CASE: $N \cap A = \prod_{f \in \mathfrak{P}} \nabla_f$ and $N \cap B = \prod_{g \in \mathfrak{Q}} \nabla_g$ for some partitions $\mathfrak{P}$ and $\mathfrak{Q}$ of the set $\{1, 2, \dots, k\}$. Then $|N \cap A| = |N_1|^r$, where $rs = k$ and $|N \cap B| = |N_1|^l$, where $lt = k$. By Lemma 3.9 it follows that n^k divides $|\operatorname{Out} N_1|^k n^{r+l} k!$. Let $l \leq r$. Then n^{rs-2r} divides $|\operatorname{Out} N_1|^k k!$, where $k = rs$. As above this implies that $rs - 2r \leq rs/(p-1)$ for every prime $p \geq 5$ such that

$p \in \pi(N_1) - \{2, 3\}$. Hence

$$s\left(1 - \frac{1}{p-1}\right) < 2 \quad \text{and so} \quad s < 2\left(\frac{p-1}{p-2}\right).$$

Since $p \geq 5$ it follows that $s \leq 2$. In each case the subgroup D fixes each N_i. Let $d \neq 1$ be an element of prime order in D. It follows from a well-known theorem of J. Thompson that $C_{N_i}(d) = C_i \neq 1$ for each $i = 1, 2, \ldots, k$. (Recall that $N \cap A$ is a product of full diagonal subgroups ∇_f with respect to the partition $\mathfrak{P}$.)

Since $C_N(d)$ contains the subgroup $< N \cap A, C_1, C_2, \ldots, C_k >$ it follows from Lemma 3.13 that $C_N(d) \geq N$ and hence $d = 1$, a contradiction.

2. SECOND CASE: $N \cap A = \prod_{f \in \mathfrak{P}} \nabla_f$ and $N \cap B = \prod_{i=1}^{k}(N_i \cap B)$, where ∇_f is a full diagonal subgroup with respect to f and the $N_i \cap B$ are conjugate to $N_1 \cap B$.

Since $O(A) \cap N = 1 = O(B) \cap N$ we have in each case that $[O(A), N \cap A] = 1 = [O(B), N \cap B]$. In this case also $[D, N \cap A] = [D, N \cap B] = 1$. By Lemma 3.13 we see that $C_G(N)$ contains D, and this leads to a contradiction.

3. THIRD CASE: $N \cap A = \prod_{i=1}^{k}(N_i \cap A)$ and $N \cap B = \prod_{i=1}^{k}(N_i \cap B)$, where $N_i \cap A \simeq N_1 \cap A$ and $N_i \cap B \simeq N_1 \cap B$.

By Lemmas 3.9 and 3.4 it follows that $|N_1|$ divides $|\operatorname{Out} N_1||N_1 \cap A||N_1 \cap B|$. A case-by-case analysis of the groups N_1 in $\mathfrak{M}$ and Lemma 2.5 shows that this also leads to a contradiction.

LEMMA 3.18: *$|N_1|$ divides $|N_1 \cap A||N_1 \cap B||\operatorname{Out} N_1|$*

Proof: Since $O(B) \cap N_1 \neq 1$ or $O(A) \cap N_1 \neq 1$ by Lemma 3.17, we may suppose that $O(B) \cap N_1 \neq 1$. In this case $N \cap B = \prod_{i=1}^{k}(N_i \cap B)$, where $N_i \cap B \simeq N_1 \cap B$ for each $i \leq k$. If $N \cap A = \prod_{i=1}^{k}(N_i \cap A)$, then by Lemmas 3.9 and 3.4 we are done. Hence $N \cap A \neq \prod_{i=1}^{k}(N_i \cap A)$. By Lemma 3.8 in this case there is a partition $\mathfrak{P}$ of the set $\{1, 2, \ldots, k\}$ such that $N \cap A = \prod_{f \in \mathfrak{P}} \nabla_f$, where ∇_f is a full diagonal subgroup with respect to f. By Lemma 3.9 we have $k = rs$ and n^k divides $|\operatorname{Out} N_1|^k |N_1 \cap B|^k n^r k!$. If $|N_1 \cap B| \equiv 0 \bmod 2$, then D fixes each N_i and, as in the proof of Lemma 3.17, we obtain a contradiction using Lemma 3.13. Therefore we may suppose that $|N_1 \cap B| \equiv 1 \bmod 2$. If 2^t is the highest power of 2 dividing the order of N_1, then $(2^t)^{rs-r}$ divides $2^{lk}2^{rs-1}$ where 2^l is the highest power of 2 dividing $|\operatorname{Out} N_1|$.

If $N_1 \simeq U_3(8)$ then $t = 6$, $l = 1$ and $6(rs - r) \leq 2rs - 1$. Hence $4rs \leq 6r$ and so $s \leq 6/4$ which leads to a contradiction. In a similar way one excludes the group $L_4(2)$.

If $N_1 \simeq M_{11}$ then $t = 4, l = 1$ and $4(rs - r) \leq 2rs - 1$, hence $2rs \leq 4r$ and $s \leq 2$. In this case $D \subseteq G_0$ and we obtain a contradiction as in the proof of Lemma 3.17. In a similar way the groups $\mathrm{PSp}_4(3)$ and $L_3(3)$ are excluded.

If $N_1 \simeq L_3(2)$, then $(2^3.3.7)^{rs-r}$ divides $2^{rs}|N_1 \cap B|^{rs}rs!$. Hence $3(rs - r) \leq 2rs - 1$ and so $rs < 3r$ and $s < 3$, which also leads to a contradiction.

It remains to consider the groups $L_2(q)$ with odd q. Since the only subgroup of $L_2(q)$ which is maximal in $L_2(q)$ and has odd order is a group of order $q(q-1)/2$ where $q \equiv 3 \bmod 4$, then $(\frac{q-1}{2}, q+1) = 1$ and $(q+1)^{rs-r}$ divides $|\operatorname{Out} N_1|^{sr}(rs)!$. It is easy to see that this also leads to a contradiction.

Lemma 3.19: $D \subseteq G_0$.

Proof: Assume that $D \not\subseteq G_0$. Let $|N_i \cap A| \equiv 0 \bmod 2$. Since $[O(A), N \cap A] \subseteq O(A) \cap N$ and has odd order, then if a is an element of $O(A)$ and τ is an involution in $N_i \cap A$ we have that $\tau^a \tau$ has odd order. If $(N_i \cap A)^a = N_j \cap A$ (where $I \neq j$) then $\tau^a\tau$ has even order. Hence if $|N_i \cap A| \equiv 0 \bmod 2$, then $(N_i \cap A)^a = N_i \cap A$ for each a in $O(A)$ and each $i \in \{1, 2, \ldots, k\}$. Then $N_i \cap N_i^a$ contains $N_i \cap A$ and $N_i = N_i^a$ for each $i \in \{1, 2, \ldots, k\}$ and a in G_0. Now $|N_i \cap A| \equiv 1 \bmod 2$ and similarly $|N_i \cap B| \equiv 1 \bmod 2$. Since by Lemma 3.18, $|N_1|$ divides $|N_1 \cap A||N_1 \cap B||\operatorname{Out} N_1|$ and $N_1 \cap A$ and $N_1 \cap B$ have both odd orders, then the order of a Sylow-2-subgroup of G must divide the order of $\operatorname{Out} N_1$. It is easy to see that this is not the case. Thus D must be contained in G_0.

Lemma 3.20: *There exists no counterexample.*

If N_1 is one of the following groups: $L_4(2)$, M_{11}, $\mathrm{PSp}_4(3)$, $L_2(q)$, $L_3(3)$, $L_3(5)$, then $|\operatorname{Out}_G(N_1)|$ is a power of 2. Therefore these groups cannot occur in N. Note that in each case $|G_0 \colon N|$ divides $|\operatorname{Out}_G(N_1)|^k$.

Therefore the remaining possibilities for N_1 are: $U_3(8), L_3(8)$.

Let, for instance, $N_1 \simeq U_3(8)$. Then it follows from Lemma 3.18 that $|N_1|$ divides $|N_1 \cap A||N_1 \cap B||\operatorname{Out} N_1|$. It is well-known that the only maximal subgroup of N_1 containing a subgroup of order 19 is a subgroup $N_1 \cap A$ of order 57. Hence the other subgroup $N_1 \cap B$ must contain a Sylow-2-subgroup of order at least 2^5. It follows that it is contained in a Borel subgroup of N_1. Since this subgroup is 2-closed and is contained in the normalizer of some non-trivial subgroup of odd order, this leads to a contradiction (see, for example, (1.18) in [7]).

The other case is treated similarly. The theorem is proved.

4. Examples

Let $\hat{G} = \mathrm{SL}_2(q^m)$ where q is a prime power, m and q are odd numbers. There exists an automorphism ϕ of the field $\mathrm{GF}(q^m)$ of order m which maps an element $\alpha \in \mathrm{GF}(q)$ onto α^q. This automorphism induces an obvious automorphism of the group $\tilde{G}$ which maps every entry of a matrix in $\hat{G}$ onto its q-th power. Then $\hat{\Gamma} =< \phi > \ltimes \hat{G}$ is a subgroup of the holomorph of the group $\hat{G}$ (we identify $\hat{G}$ with its image).

Clearly the subgroup $\hat{B}$ of matrices whose first row consists of α and β and whose second row is 0 and $1/\alpha$ (where $\alpha \in \mathrm{GF}(q^m)^*$ and $\beta \in \mathrm{GF}(q^m)$) is invariant under $< \phi >$. Hence there exists a subgroup $\tilde{B} =< \phi > \ltimes B$ of order $q^m(q^m-1)m$. Let $G = \hat{G}/Z(\hat{G})$, where $Z(\hat{G})$ is the set $\{+I, -I\}$. Let $q \equiv 3 \bmod 4$. Then the image of $\tilde{B}$ in G is a subgroup B of order $q^m(q^m-1)/2$; the image of $\tilde{B}$ is a subgroup H of odd order $|B|m$.

By the theorem of Dickson the group G contains a dihedral subgroup D of order $q^m + 1$ (see Lemma 2.1). Obviously we have $BD = G$, since

$$\left(q^m + 1, \frac{q^m(q^m-1)}{2}\right) = 1$$

and the order of G is $q^m(q^{2m}-1)/2$. There exists a unique class of dihedral subgroups of order $q^m + 1$ in G. Since G is normal in the group $\Gamma = \hat{\Gamma}/Z(\hat{\Gamma})$, an application of the Frattini argument yields that $\Gamma = GN_\Gamma(D) = BN_\Gamma(D) = HN_\Gamma(D)$. Since $N_G(D) = D$, it follows that $K = N_\Gamma(D)$ is an extension of a group D by a cyclic group of order m. This implies that K is a 2-nilpotent group of order $m(q^m+1)$. Since $\Gamma = HK$, where the order of Γ is $m(q^m(q^{2m}-1))/2$, the order of H is $m(q^m(q^m-1))/2$ and the order of K is $m(q^m+1)$. Then the order of $H \cap K$ is m, $H = O(H)$ and $H \cap K = O(H) \cap O(K)$ is not contained in $O(\Gamma)$.

A similar example can be constructed if $q \equiv 1 \bmod 4$ and $G \simeq \mathrm{PGL}(2, q)$.

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